CALCULATION OF THE TEMPERATURE FIELD IN PLANE SLOTTED CHANNELS WITH LAMINAR GAS FLOW

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The solution is presented for the temperature fields at the wall and in the laminar gas stream. The system is steady-state, and the heat flux to the wall is constant.

Reference [1] has reported an investigation of the temperature field with laminar flow of a gas between parallel flat walls allowing for the heat arising from internal friction and from work done by pressure forces. The wall is assumed to be thermally thin.

In some situations the heat due to internal friction and pressure forces may be neglected, but the two-dimensional nature of the wall must be taken into account.

The present paper gives a solution to the problem of computing the temperature field with laminar flow of a gas between parallel walls. The problem postulated is illustrated in Fig. 1.

The problem has been solved under the following premises: a) the temperature profile at the entrance is uniform; b) the velocity profile at the entrance is parabolic; c) the physical properties of the gas and of the wall materials are independent of temperature; d) we neglect thermal conduction of the gas along the streamwise direction; e) we neglect friction heating; f) the channel is quite wide; g) the heat flux to the inside of the walls is constant.

In view of the symmetry we shall consider only one half of the channel.

For the wall we have the equation

$$\frac{\partial^2 T_w}{\partial x^2} + \frac{\partial^2 T_w}{\partial y^2} = 0 \tag{1}$$

with boundary conditions

$$q = -\lambda_w \frac{\partial T_w}{\partial y} = \text{const} \quad \text{for} \quad y = b + c;$$

$$\frac{\partial T_w}{\partial x} = 0 \quad \text{for} \quad x = 0, \quad x = l;$$

$$f(x) = -\lambda_w \frac{\partial T_w}{\partial y} \quad \text{for} \quad y = b.$$

In view of conditions b), c), d) and f), the energy equation for the gas stream has the form

$$\lambda_r \frac{\partial^2 T_g}{\partial y^2} = c_p \rho u_x \frac{\partial T_g}{\partial x}, \tag{2}$$

with boundary conditions

$$T_g = T_0 \quad \text{for} \quad x = 0;$$

$$\frac{\partial T_g}{\partial y} = 0 \quad \text{for} \quad y = 0;$$

$$f(x) = -\lambda_g \frac{\partial T_g}{\partial y} \quad \text{for} \quad y = b.$$

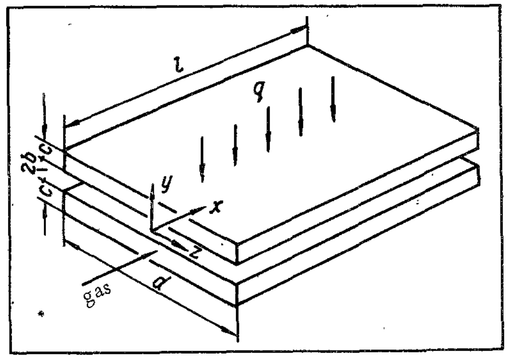

Fig. 1. Schematic of the problem.

The solution of the second boundary value problem for a rectangle has been given in [2] and may be written in the form

$$T_w = \sum_{n=1}^{\infty} (\alpha_n \operatorname{sh} \lambda_n y + \beta_n \operatorname{ch} \lambda_n y) \cos \lambda_n x + (\alpha_0 y + \beta_0). \tag{3}$$

Differentiating (3) and substituting into the boundary conditions, we have, on the side y = b + c of the rectangle

$$\alpha_0 = -q/\lambda_w \quad \text{for} \quad n = 0, \tag{4}$$

$$\lambda_w \lambda_n [\alpha_n \operatorname{ch} \lambda_n (b + c) + \beta_n \operatorname{sh} \lambda_n (b + c)] = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{5}$$

On the side y = b, taking the function $f(x)$ as known and expanding it in a Fourier series, we obtain

$$\alpha_0 = -f_0/2\lambda_w \quad \text{for} \quad n = 0, \tag{6}$$

$$-\lambda_w \lambda_n (\alpha_n \operatorname{ch} \lambda_n b + \beta_n \operatorname{sh} \lambda_n b) = f_n \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{7}$$

From (4) and (6) we find

$$f_0 = 2q.$$

Solving (5) and (7) we find

$$\alpha_n = -\frac{f_n}{\lambda_w \lambda_n} \frac{\operatorname{sh} \lambda_n (b + c)}{\operatorname{sh} \lambda_n c},$$

$$\beta_n = \frac{f_n}{\lambda_w \lambda_n} \frac{\operatorname{ch} \lambda_n (b + c)}{\operatorname{sh} \lambda_n c}.$$

Substituting the values of α_0, α_n, β_n into (3) we obtain

$$T_w = \beta_0 - \frac{q}{\lambda_w} y + \frac{1}{\lambda_w} \sum_{n=1}^{\infty} f_n \frac{1}{\lambda_n} \frac{\operatorname{ch} \lambda_n (b + c - y)}{\operatorname{sh} \lambda_n c} \cos \lambda_n x. \tag{8}$$

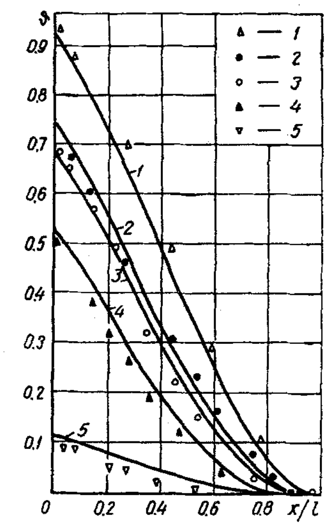

Fig. 2. Comparison of the results of calculation of wall temperatures with experimental data: 1) for $K_1 = 1.79 \cdot 10^5$ and $K_2 = 2.84 \cdot 10^{-2}$; 2) $1.45 \cdot 10^6$ and $2.44 \cdot 10^{-3}$; 3) $2.22 \cdot 10^5$ and $5.80 \cdot 10^{-4}$; 4) $1.21 \cdot 10^6$ and $5.58 \cdot 10^{-4}$; 5) $1.11 \cdot 10^6$ and $7.42 \cdot 10^{-5}$.

We shall seek a solution of the energy equation (2) in the form

$$T_g = \theta_b(x) + \theta(x, y), \tag{9}$$

where $\theta_b(x)$ is the gas temperature at $y = b$.

Differentiating (9) and substituting the result into (2) we obtain

$$\frac{\partial^2 \theta(x, y)}{\partial y^2} = K_1 \left(1 - \frac{y^2}{b^2}\right)\left[\frac{\partial \theta(x, y)}{\partial x} + \frac{\partial \theta_b(x)}{\partial x}\right], \tag{10}$$

with the boundary conditions

$$\theta(x, y) = T_0 - \theta_b(0) \quad \text{for } x = 0;$$
$$\frac{\partial \theta(x, y)}{\partial y} = 0 \quad \text{for } y = 0;$$
$$\theta(x, y) = 0 \quad \text{for } y = b.$$

Without a prior assumption that the function T_g is continuous at the point $(0, b)$, we may write

$$T_g(0, b) - \theta_b(0) \equiv \Delta T \neq 0.$$

We shall seek the function $\theta(x, y)$ as the series

$$\theta(x, y) = \sum_{n=1}^{\infty} P_n(x) V_n(y), \tag{11}$$

where $V(y)$ are eigenfunctions of the boundary value problem

$$V_n''(y) + \nu_n^2 K_1 (1 - y^2/b^2) V_n(y) = 0,$$
$$V_n'(0) = 0, \quad V_n(b) = 0. \tag{12}$$

Using (11) and (12), and dividing by $K_1(1 - y^2/b^2)$, we obtain

$$\sum_{n=1}^{\infty} [P_n'(x) + \nu_n^2 P_n(x)] V_n(y) + \theta_b'(x) = 0. \tag{13}$$

We expand unity in a series with respect to the eigenfunctions $V_n(y)$:

$$1 = \sum_{n=1}^{\infty} c_n V_n(y).$$

Then

$$\sum_{n=1}^{\infty} [P_n'(x) + \nu_n^2 P_n(x) + c_n \theta_b'(x)] V_n(y) = 0. \tag{14}$$

The solution of the differential equation in the brackets under the initial conditions

$$\theta(x, y) = \Delta T \quad \text{for } x = 0,$$
$$\sum_{n=1}^{\infty} P_n(0) V_n(y) = \Delta T \equiv \sum_{n=1}^{\infty} \Delta T c_n V_n(y),$$
$$P_n(0) = \Delta T c_n,$$

has the form

$$P_n(x) = \Delta T c_n \times \exp - \nu_n^2 x - c_n \int_0^x \exp - \nu_n^2 (x - \xi) \theta_b'(\xi) \, d\xi. \tag{15}$$

Substituting (15) into (11) and then (9), we obtain

$$T_g = \theta_b(x) + \sum_{n=1}^{\infty} c_n \left[\Delta T \exp - \nu_n^2 x - \int_0^x \exp - \nu_n^2 (x - \xi) \theta_b'(\xi) \, d\xi \right] V_n(y). \tag{16}$$

MATCHING CONDITIONS

At the gas-wall boundary we have

$$T_g = T_w, \quad \lambda_g \frac{\partial T_g}{\partial y} = \lambda_w \frac{\partial T_w}{\partial y}.$$

Using (8) and (6) we obtain

$$\theta_b(x) = - \frac{q}{\lambda_w} b + \beta_0 + \frac{1}{\lambda_w} \sum_{n=1}^{\infty} f_n \left(\frac{1}{\lambda_n} \operatorname{cth} \lambda_n c\right) \cos \lambda_n x,$$

$$f(x) = - \lambda_g \sum_{n=1}^{\infty} c_n \left[\Delta T \exp - \nu_n^2 x - \int_0^x \exp - \nu_n^2 (x - \xi) \theta_b'(\xi) \, d\xi\right] V_n'(b). \tag{17}$$

Differentiating the first equation of (17) and substituting it into the second, and replacing n by k, we obtain

$$f(x) + \lambda_g \sum_{n=1}^{\infty} c_n V_n'(b) [\Delta T \exp - \nu_n^2 x + \frac{1}{\lambda_w} \sum_{k=1}^{\infty} f_k \operatorname{cth} \lambda_k c \times \int_0^x \exp - \nu_n^2 (x - \xi) \sin \lambda_k \xi \, d\xi] = 0. \tag{18}$$

To determine the unknown quantities ΔT, f_1, f_2, ..., f_n we shall set up a system of equations, for which we shall multiply (18) by 1, $\cos \lambda_1 x$, $\cos \lambda_2 x$, ..., $\cos \lambda_n x$ and integrate over the range $(0, l)$. Because of the orthogonality we obtain

$$ql + \Delta T\left[\lambda_g \sum_{n=1}^{\infty} c_n V'_n(b)\frac{1-\exp - \nu_n^2 l}{\nu_n^2}\right] + K_2 \sum_{k=1}^{\infty} f_k \operatorname{cth}\lambda_k c\left\{\left[\sum_{n=1}^{\infty}\frac{\nu_n^2 c_n V'_n(b)}{\nu_n^4+\lambda_k^2}\right]\frac{1-(-1)^k}{\lambda_k} + \left[\sum_{n=1}^{\infty}\frac{\lambda_k c_n V'_n(b)}{\nu_n^4+\lambda_k^2}\right]\frac{1-\exp - \nu_n^2 l}{\nu_n^2}\right\} = 0 \quad (19)$$

for m = 0, and

$$\frac{l}{2} f_m + \Delta T\left\{\lambda_r \sum_{n=1}^{\infty}\frac{\nu_n^2 c_n V'_n(b)}{\nu_n^4+\lambda_m^2}\left[1-(-1)^m \exp - \nu_n^2 l\right]\right\} + \sum_{k=1}^{\infty} f_k (K_2 \operatorname{cth}\lambda_k c)\left\{\left[\sum_{n=1}^{\infty}\frac{\nu_n^2 c_n V'_n(b)}{\nu_n^4+\lambda_k^2}\right]\frac{\lambda_k}{\lambda_k^2-\lambda_m^2}\times\right.$$

$$\times\left[1-(-1)^{k+m}\right] + \lambda_k\left[\sum_{n=1}^{\infty}\frac{c_n V'_n(b)}{\nu_n^4+\lambda_k^2}\right]\frac{\nu_n^2}{\nu_n^4+\lambda_m^2}\times$$

$$\left.\times\left[1-(-1)^m \exp - \nu_n^2 l\right]\right\} + f_m (K_2 \operatorname{cth}\lambda_m c)\times$$

$$\times\left\{-\lambda_m\left[\sum_{n=1}^{\infty}\frac{c_n V'_n(b)}{\nu_n^4+\lambda_m^2}\right]\frac{l}{2} + \lambda_m\left[\sum_{n=1}^{\infty}\frac{c_n V'_n(b)}{\nu_n^4+\lambda_m^2}\right]\times\right.$$

$$\left.\times\frac{\nu_n^2\left[1-(-1)^m \exp - \nu_n^2 l\right]}{\nu_n^4+\lambda_m^2}\right\} = 0 \quad (20)$$

with m = 1, 2, 3, ...

Putting $\tau = y/b$ in (12), we obtain

$$Q''_n + \mu_n^2(1-\tau^2)Q_n = 0, \quad Q'_n(0) = 0, \quad Q_n(1) = 0.$$

$$V_n(y) = Q_n(\tau), \quad \mu_n^2 = \nu_n^2 b^2 K_1. \quad (21)$$

The eigenfunctions of the boundary problem (21) in the interval (0, 1) are orthogonal and are normalized with weight $(1 - \tau^2)$:

$$[Q_n, Q_m] = \int_0^1 (1-\tau^2) Q_n(\tau) Q_m(\tau)\, d\tau = \delta_{nm}. \quad (22)$$

We shall expand unity in a series with respect to eigenfunctions $Q_n(\tau)$:

$$1 = \sum_{n=1}^{\infty} c_n Q_n(\tau),$$

where

$$c_n = \int_0^1 (1-\tau^2) Q_n(\tau)\, d\tau. \quad (23)$$

Integrating (21) and taking into account that $Q'_n(0) = 0$, we find

$$\mu_n^2 \int_0^1 (1-\tau^2) Q_n(\tau)\, d\tau = -\mu_n^2 c_n.$$

Since

$$V'_n(y) = \frac{1}{b} Q'_n(\tau),$$

then

$$c_n V'_n(b) = -c_n \mu_n^2/b. \quad (24)$$

In addition we have that

$$\sum_{n=1}^{\infty} c_n^2 = \frac{2}{3}. \quad (25)$$

Substituting the values of (24) and (25) into (19) and (20), and designating $a_k = 1/b^2 K_1 \lambda_k^2$, we obtain an infinite system of equations with respect to ΔT, f_n:

$$\Delta T\left(\frac{2}{3} - \sum_{n=1}^{\infty} c_n^2 \exp - \nu_n^2 l\right) + \frac{1}{\lambda_w}\sum_{n=1}^{\infty} f_k \frac{\operatorname{cth}\lambda_k c}{\lambda_k}\times\left\{\frac{2}{3}\left[1-(-1)^k\right] + \sum_{n=1}^{\infty}\frac{c_n^2\left[(-1)^k - \exp - \nu_n^2 l\right]}{1+a_k\mu_n^4}\right\} = -\frac{ql}{\lambda_g b K_1} \quad (26)$$

for m = 0;

$$\Delta T \lambda_g b K_1\left\{\frac{2}{3} - \sum_{n=1}^{\infty} c_n^2\left[\frac{1-(-1)^m \exp - \nu_n^2 l}{1+a_m\mu_n^4} + (-1)^m \exp - \nu_n^2 l\right]\right\} + K_1 K_2 b \sum_{k=1}^{\infty} f_k (\lambda_k \operatorname{cth}\lambda_k c)\times$$

$$\times\left\{\frac{1-(-1)^{k+m}}{l\,(\lambda_k^2-\lambda_m^2)}\left[\frac{2}{3} - \sum_{n=1}^{\infty}\frac{c_n^2}{1+a_k\mu_n^4}\right] + \frac{1}{\lambda_m^2}\left[\sum_{n=1}^{\infty} c_n^2 \frac{a_k\mu_n^4}{1+a_k\mu_n^4}\,\frac{1-(-1)^m \exp - \nu_n^2 l}{1+a_m\mu_n^4}\right]\right\} +$$

$$+ f_m\left\{K_1 K_2 b \frac{\operatorname{cth}\lambda_m c}{\lambda_m}\left[\sum_{n=1}^{\infty} c_n^2 \frac{a_m\mu_n^4}{1+a_m\mu_n^4}\times\frac{1-(-1)^m \exp - \nu_n^2 l}{1+a_m\mu_n^4}\right] - K_2 \frac{l}{2b}\,\frac{\operatorname{cth}\lambda_m c}{\lambda_m}\left(\sum_{n=1}^{\infty}\frac{c_n^2\mu_n^2}{1+a_m\mu_n^4}\right) - \frac{l}{2}\right\} = 0 \quad (27)$$

for m = 1, 2, 3, ...

The system of Eqs. (26) and (27) was limited to m < 30 and was solved on a BESM-2M computer.

Tests were conducted by the author in plane-slotted channels of height 2b = 0.2, 0.4 and 0.8 mm, relative width d/d_e = 25, and relative length l/d_e = 125.

The channel walls were made of materials with thermal conductivity ranging from 1 to 390 W/m · deg.

The results of the calculations are shown in Fig. 2 as the solid lines, the experimental data being the points.

The wall temperature at the section entrance differed from that of the gas both in the experiments and in the calculations.

NOTATION

$2b$ is the distance between the walls; c is the wall thickness; l is the channel length; x/l is the relative channel length; d is the width of the experimental channel; d_e is the equivalent diameter; $d_e = 4b$; T_g is the gas temperature; T_0 is the gas temperature at the entrance; T_w is the wall temperature; ϑ is the dimensionless wall temperature, $\vartheta = (T_w - T)/(T_w - T_0)$; λ_g is the thermal conductivity of the gas; λ_w is the thermal conductivity of the wall; u_x is the gas velocity; u_m is the mean velocity of the gas in the channel; c_p is the specific heat of the gas; ρ is the gas density; q is the specific heat flux; $K_1 = (3/8)(1/b)\,\mathrm{Re}\,\mathrm{Pr}$; $K_2 = \lambda_g/\lambda_w$; $\mathrm{Re} = 4\,u_m \rho b/\mu$; $\mathrm{Pr} = c_p \mu/\lambda_g$.

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